**Year 11 Mathematics Extension II Term 4 Assessment 2002**

**Question 1: (15 Marks) (START A NEW PAGE)**

(a) Given $z = 3 + 2i$ and $w = 1 + 3i$. Express the following in the form $z = a + ib$ where $a$ and $b$ are real numbers

(i) $z + \overline{w}$ **2**

(ii) $z^2$ **2**

(iii) $\frac{z}{w}$ **3**

(b) (i) Express $1 + i\sqrt{3}$ in mod-arg form. **2**

(ii) Simplify $\left(1 + i\sqrt{3}\right)^9$. **2**

(c) (i) Show that $z = i$ is a root of the quadratic equation $(2 - i)z^2 - (1 + i)z + 1 = 0$. **1**

(ii) Find the other root of the above equation in the form $z = c + id$ where $c$ and $d$ are real numbers. **3**

**Question 2: (15 Marks) (START A NEW PAGE)**

(a) Find real values for $x$ and $y$ if $(x + iy)(3 + i) = 5i$ **2**

(b) Make neat sketches of the following loci on separate diagrams.

(i) $|z| = |z - 4 - 2i|$. **2**

(ii) $\arg(z - i) = \frac{\pi}{4}$. **2**

(iii) $|z - 3 + 4i| \leq 5$. **3**

(c) Let $z = x + iy$ be any non-zero complex number.

(i) Express $z + \frac{1}{z}$ in the form $p + iq$. **2**

(ii) Given that $z + \frac{1}{z}$ is real, show that $y = 0$ or $x^2 + y^2 = 1$. **2**

(iii) If $y = 0$, show that $\left|z + \frac{1}{z}\right| \geq 2$. **2**

**Question 3: (15 Marks) (START A NEW PAGE)**

(a) (i) Find both square roots of $z = 5 - 12i$ , writing your answers in the form $z = x + iy$ where $x$ and $y$ are real numbers. **2**

(ii) Hence solve $z^2 + z - 1 + 3i = 0$ writing your answers in the form $z = p + iq$ where $p$ and $q$ are real numbers. **3**

(b) (i) Express $1 + i$ in mod-arg form **2**

(ii) Find the smallest <u>positive</u> value of $m$ if $(1+i)^m = (1-i)^m$. **3**

(c) $z_1$ , $z_2$ and $z_3$ are the three cube roots of a complex number $z$ and $|z| = 1$.

(i) Assuming that $\arg(z_1) = \theta$ where $0 \le \theta \le \frac{\pi}{2}$ , draw a diagram clearly showing the positions of the complex numbers $z_1$ , $z_2$ and $z_3$. **2**

(ii) Evaluate $|z_1 - 1|^2 + |z_2 - 1|^2 + |z_3 - 1|^2$ . **3**

**Question 4: (15 Marks) (START A NEW PAGE)**

(a) (i) Given $z_1 = \cos\alpha + i\sin\alpha$ , show that $1 + z_1 = 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$. **2**

(ii) If $z_2 = \cos\beta + i\sin\beta$ find, in terms of $\alpha$ and $\beta$ , the modulus and argument of the complex number $\dfrac{z_2 + 1}{z_1 + 1}$ . **2**

(iii) For two complex numbers $z_1$ $(\neq -1)$ and $z_2$ $(\neq -1)$, $|z_1| = |z_2| = 1$ and $\dfrac{z_2 + 1}{z_1 + 1} = i$. Find $z_1$ and $z_2$ in the form $z = x + iy$ where $x$ and $y$ are real numbers. **4**

(b) (i) Find the nine roots of $z^9 - 1 = 0$, leaving your answers in mod-arg form. **2**

(ii) If $P(z) = z^6 + z^3 + 1$, explain why the roots of $P(z) = 0$ are also roots of $z^9 - 1 = 0$. **1**

(iii) Show the position of the roots of $P(z) = 0$ on an Argand Diagram. **2**

(iv) Express $P(z)$ in the form $P(z) = (z^2 - az + 1)(z^2 - bz + 1)(z^2 - cz + 1)$ where $a$, $b$ and $c$ are real numbers. **2**

# THIS IS THE END

**Year 11 Term 4 Assessment – Extension II – 2002**

**Question 1:**

1(a) (i)

$$z + \overline{w} = (3+2i) + (1-3i)$$
$$= 4 - i$$

(ii)

$$z^2 = (3+2i)^2$$
$$= 9 + 12i + 4i^2$$
$$= 5 + 12i$$

(iii)

$$\frac{z}{w} = \frac{3+2i}{1+3i} \times \frac{1-3i}{1-3i}$$
$$= \frac{3 - 9i + 2i - 6i^2}{1+9}$$
$$= \frac{9-7i}{10}$$
$$= \frac{9}{10} - \frac{7}{10}i$$

1(b) (i)

$$\tan\theta = \sqrt{3}$$
$$\arg(z) = \theta = \frac{\pi}{3}$$
$$|z| = \sqrt{1^2 + (\sqrt{3})^2}$$
$$= 2$$
$$1 + i\sqrt{3} = 2cis\frac{\pi}{3}$$

(ii)

$$(1+i\sqrt{3})^9 = \left(2cis\frac{\pi}{3}\right)^9$$
$$= 2^9 cis 3\pi$$
$$= 512(\cos 3\pi + i\sin 3\pi)$$
$$= -512$$

1(c) (i)

$$P(i) = (2-i)(i)^2 - (1+i)(i) + 1$$
$$= (2-i)(-1) - (1+i)(i) + 1$$
$$= -2 + i - i - i^2 + 1$$
$$= -2 + i - i + 1 + 1$$
$$= 0$$

$\therefore z = i$ is a root

(ii) Let other root be $z = \alpha$

$$\alpha + i = \frac{1+i}{2-i} \qquad \left(\text{sum of roots} = -\frac{b}{a}\right)$$

$$\alpha = \frac{1+i}{2-i} - i$$
$$= \frac{1+i-2i+i^2}{2-i}$$
$$= \frac{-i}{2-i} \times \frac{2+i}{2+i}$$
$$= \frac{-2i-i^2}{5}$$
$$= \frac{1}{5} - \frac{2}{5}i$$

**Question 2:**

2(a)

$$(x+iy)(3+i)=5i$$

$$3x+ix+3iy+i^2y=5i$$

$$(3x-y)+(x+3y)i=5i$$

$$3x-y=0\text{............}(1)$$

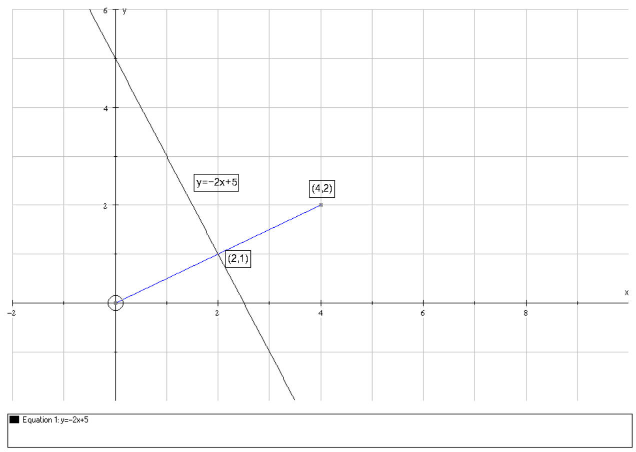
$$x+3y=5\text{...........}(2)$$

from (1) $\quad y=3x$

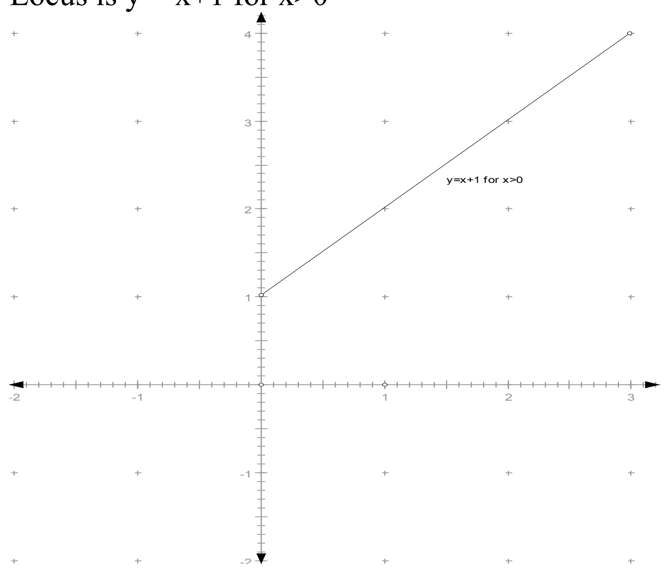
sub into (2) $\quad x+9x=5$

$$\therefore \quad x=\frac{1}{2} \text{ and } y=\frac{3}{2}$$

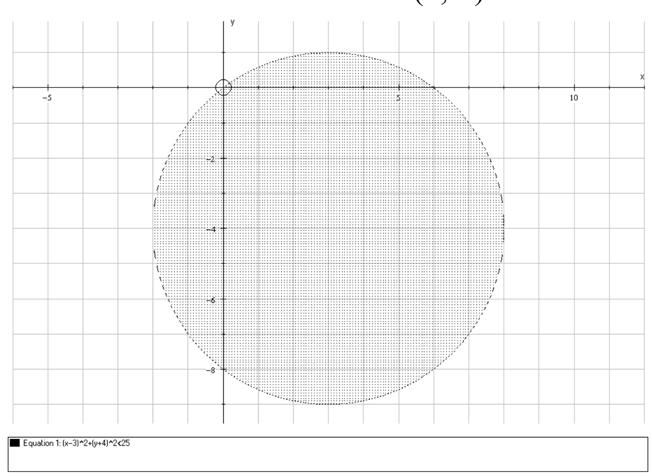
2(b) (i) Locus is y = -2x+5




(ii) Locus is y = x+1 for x>0




(iii) Locus is interior of circle center (3,-4) and radius = 5



2(c) (i)

$$z+\frac{1}{z}=(x+iy)+\frac{1}{x+iy}$$
$$=x+iy+\frac{x-iy}{x^2+y^2}$$
$$=\frac{x(x^2+y^2)+iy(x^2+y^2)+x-iy}{x^2+y^2}$$
$$=\frac{x(x^2+y^2+1)}{x^2+y^2}+\frac{y(x^2+y^2-1)}{x^2+y^2}i$$

(ii) If $z+\frac{1}{z}$ is real then Im$(z)=0$

$\therefore\ y(x^2+y^2-1)=0$

$\therefore\ y=0$ or $x^2+y^2=1$

(iii) If $y=0$ then $z+\frac{1}{z}=\frac{x(x^2+1)}{x^2}$

$$=x+\frac{1}{x}$$

$$\therefore\ \left|z+\frac{1}{z}\right|=\left|x+\frac{1}{x}\right|$$
$$=\left|x+\frac{1}{x}-2+2\right|$$
$$=\left|\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2+2\right| \text{ for } x>0$$
$$\geq 2 \quad \text{since } \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2\geq 0$$

by symmetry, result is also true for $x<0$

**Question 3:**

3(a) (i) Let $z = a + ib$

$(a+ib)^2 = 5 - 12i$

$\therefore a^2 - b^2 + 2iab = 5 - 12i$

equating terms

$a^2 - b^2 = 5 \ldots\ldots\ldots\ldots(1)$

$ab = -6 \ldots\ldots\ldots\ldots\ldots\ldots(2)$

by inspection $a = 3, b = -2$ or $a = -3, b = 2$

$\therefore z = 3 - 2i$ or $z = -3 + 2i$

(ii) $z = \dfrac{-1 \pm \sqrt{1 - 4(-1+3i)}}{2}$

$= \dfrac{-1 \pm \sqrt{5 - 12i}}{2}$

$= \dfrac{-1 + (3-2i)}{2}$ or $\dfrac{-1 - (3-2i)}{2}$

$z = 1 - i$ or $-2 + i$

3(b) (i) $1 + i = \sqrt{2}cis\left(\dfrac{\pi}{4}\right)$

(ii) $1 - i = \sqrt{2}cis\left(-\dfrac{\pi}{4}\right)$

$\therefore \left\{\sqrt{2}cis\left(\dfrac{\pi}{4}\right)\right\}^m = \left\{\sqrt{2}cis\left(-\dfrac{\pi}{4}\right)\right\}^m$

$\left\{\sqrt{2}cis\left(\dfrac{\pi}{4}\right)\right\}^m \div \left\{\sqrt{2}cis\left(-\dfrac{\pi}{4}\right)\right\}^m = 1$

$\left\{cis\left(\dfrac{m\pi}{4}\right) \div cis\left(-\dfrac{m\pi}{4}\right)\right\} = 1$

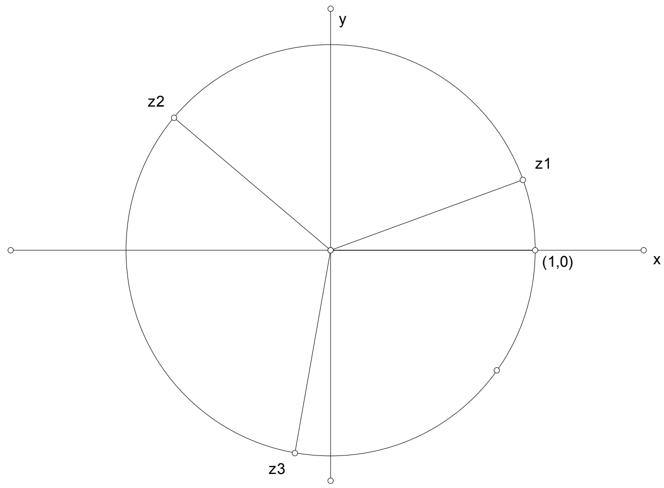
$cis\left(\dfrac{m\pi}{2}\right) = 1$

$\dfrac{m\pi}{2} = 0, 2\pi, 4\pi, 6\pi \ldots\ldots\ldots$

$m = 0, 4, 8, 12, \ldots.$

$\therefore$ smallest positive value is $m = 4$

3(c) (i)




(ii) $|z_1 - 1|^2 + |z_2 - 1|^2 + |z_3 - 1|^2$

$= (z_1 - 1)(\bar{z}_1 - 1) + (z_2 - 1)(\bar{z}_2 - 1) + (z_3 - 1)(\bar{z}_3 - 1)$

$= z_1\bar{z}_1 - (z_1 + \bar{z}_1) + 1 + z_2\bar{z}_2 - (z_2 + \bar{z}_2) + 1 + z_3\bar{z}_3 - (z_3 + \bar{z}_3) + 1$

$= 6 - (z_1 + z_2 + z_3) - (\bar{z}_1 + \bar{z}_2 + \bar{z}_3)$

$= 6$ since $z_1 + z_2 + z_3$ (and the conjugates) is the sum of the roots of $z^3 = 1$ and $\therefore = 0$

or

$|z_1 - 1|^2 + |z_2 - 1|^2 + |z_3 - 1|^2$

$= [2 + 2\cos\theta] + \left[2 + 2\cos\left(\tfrac{2\pi}{3} + \theta\right)\right] + \left[2 + 2\cos\left(\tfrac{2\pi}{3} - \theta\right)\right]$

$= 6 + 2\left[\cos\theta + \cos\left(\tfrac{2\pi}{3}\right)\cos\theta - \sin\left(\tfrac{2\pi}{3}\right)\sin\theta + \cos\left(\tfrac{2\pi}{3}\right)\cos\theta + \sin\left(\tfrac{2\pi}{3}\right)\sin\theta\right]$

$= 6 + 2\left[\cos\theta + 2\cos\left(\tfrac{2\pi}{3}\right)\cos\theta\right]$

$= 6 + 2\left[\cos\theta + 2\left(-\tfrac{1}{2}\right)\cos\theta\right]$

$= 6 + 2[\cos\theta - \cos\theta]$

$= 6$

**Question 4:**

4(a) (i) $1+z_1 = 1+\cos a + i\sin a$

$= 2\cos^2\left(\frac{a}{2}\right) + 2i\sin\left(\frac{a}{2}\right)\cos\left(\frac{a}{2}\right)$

$= 2\cos\left(\frac{a}{2}\right)\left\{\cos\frac{a}{2} + i\sin\frac{a}{2}\right\}$

(ii) $\frac{z_2+1}{z_1+1} = \frac{2\cos\frac{\beta}{2}\left(\cos\frac{\beta}{2}+i\sin\frac{\beta}{2}\right)}{2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)}$

$= \frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} cis\left(\frac{\beta-a}{2}\right)$

$\left|\frac{z_2+1}{z_1+1}\right| = \frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}$ and $\arg\left(\frac{z_2+1}{z_1+1}\right) = \frac{\beta-a}{2}$

(iii) $|i|-1 \quad \arg i = \frac{\pi}{2}$

$\therefore \quad \frac{\beta-\alpha}{2} = \frac{\pi}{2}$

$\beta = a+\pi$

$\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} = 1$

$\frac{\cos\left(\frac{\alpha}{2}+\frac{\pi}{2}\right)}{\cos\frac{\alpha}{2}} = 1$

$\frac{-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = 1$

$\tan\frac{\alpha}{2} = -1$ for $-\pi < \alpha \le \pi$

$\frac{\alpha}{2} = -\frac{\pi}{4}$

$\alpha = -\frac{\pi}{2}$ and $\beta = \frac{\pi}{2}$

$z_1 = cis\left(-\frac{\pi}{2}\right) = -i$ and $z_2 = cis\left(\frac{\pi}{2}\right) = i$

4(b) (i) $z^9 = 1$

let roots be $z = rcis\theta$

$\therefore r^9 cis9\theta = cis2k\pi,\ k = 0,1,2,3,4,5,6,7,8$

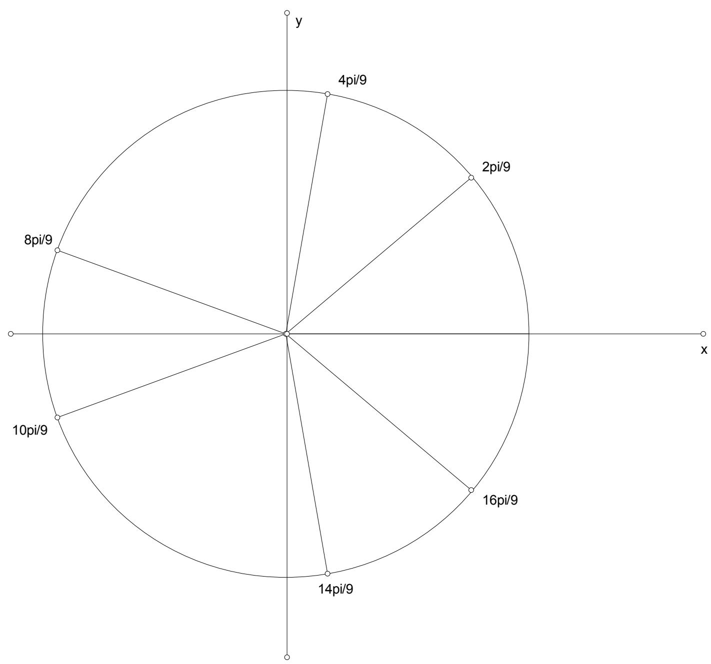
$r = 1,\quad 9\theta = 2k\pi,\ k = 0,1,2,3,4,5,6,7,8$

$z = cis0, cis\frac{2\pi}{9}, cis\frac{4\pi}{9}, cis\frac{6\pi}{9}, cis\frac{8\pi}{9}, cis\frac{10\pi}{9}, cis\frac{12\pi}{9}, cis\frac{14\pi}{9}, cis\frac{16\pi}{9}$

(ii)
$$z^9 - 1 = \left(z^3\right)^3 - 1$$
$$= \left(z^3 - 1\right)\left(\left(z^3\right)^2 + \left(z^3\right) + 1\right)$$
$$= \left(z^3 - 1\right)\left(z^6 + z^3 + 1\right)$$

since $z^6 + z^3 + 1$ is a factor of $z^9 - 1$, the roots of $z^6 + z^3 + 1$ are also roots o f $z^9 - 1$.

(iii)




(iv) The six roots of $z^6 + z^3 + 1$ come in 3 conjugate pairs

Let them be $\alpha = \frac{2\pi}{9}, \overline{\alpha} = \frac{16\pi}{9}, \beta = \frac{4\pi}{9}, \overline{\beta} = \frac{14\pi}{9}, \gamma = \frac{8\pi}{9}, \overline{\gamma} = \frac{10\pi}{9}$

$$z^6 + z^3 + 1 = \left(z^2 - 2z\cos a + |a|^2\right)\left(z^2 - 2z\cos\beta + |\beta|^2\right)\left(z^2 - 2z\cos\gamma + |\gamma|^2\right)$$
$$= \left(z^2 - 2z\cos\tfrac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\tfrac{4\pi}{9} + 1\right)\left(z^2 - 2z\cos\tfrac{8\pi}{9} + 1\right)$$